The Eccentric Digraph of a Lintang Graph

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ABSTRACT
Let $G$ be a graph with a set of vertices $V(G)$ and a set of edges $E(G)$. The distance from vertex $u$ to vertex $v$ in $G$, denoted by $d(u,v)$, is a length of the shortest path from vertex $u$ to $v$. The eccentricity of vertex $u$ in graph $G$ is the maximum distance from vertex $u$ to any other vertices in $G$, denoted by $e(u)$. Vertex $v$ is an eccentric vertex from $u$ if $d(u,v) = e(u)$. The eccentric digraph $ED(G)$ of a graph $G$ is a graph that has the same set of vertices as $G$, and there is an arc (directed edge) joining vertex $u$ to $v$ if $v$ is an eccentric vertex from $u$. Boland and Miller [1] introduced the eccentric digraph of a digraph. They also proposed an open problem to find the eccentric digraph of various classes of graphs. In this paper, we tackle this open problem for the class of lintang graph.

Key words: eccentricity, eccentric digraph, lintang graph

1. Introduction

Most of notations and terminologies follow that of Chartrand and Oellermann [3]. Let $G$ be a graph with a set of vertices $V(G)$ and a set of edges $E(G)$. The distance from vertex $u$ to vertex $v$ in $G$, denoted by $d(u,v)$, is a length of the shortest path from vertex $u$ to $v$. If there is no a path joining vertex $u$ and vertex $v$, then $d(u,v) = \infty$. The eccentricity of vertex $u$ in graph $G$ is the maximum distance from vertex $u$ to any other vertices in $G$, denoted by $e(u)$, and so $e(u) = \max \{d(u,v) | v \in V(G)\}$. Radius of a graph $G$, denoted by $rad(G)$, is the minimum eccentricity of every vertex in $G$. The diameter of a graph $G$, denoted by $diam(G)$, is the maximum eccentricity of every vertex in $G$. If $e(u) = rad(G)$, then vertex $u$ is called central vertex. Center of a graph $G$, denoted by $cen(G)$, is an induced subgraph formed from central vertices of $G$. Vertex $v$ is an eccentric vertex from $u$ if $d(u,v) = e(u)$. The eccentric digraph $ED(G)$ of a graph $G$ is a graph that has the
same set of vertices as \(G\), \(V(ED(G)) = V(G)\), and there is an arc (directed edge) joining vertex \(u\) to \(v\) if \(v\) is an eccentric vertex from \(u\). An arc of a digraph \(D\) joining vertex \(u\) to \(v\) and vertex \(v\) to \(u\) is called a symmetric arc. Further, Fred Buckley concluded that almost in every graph \(G\), its eccentric digraph is \(ED(G) = \overline{G^*}\), where \(\overline{G^*}\) is a complement of \(G\) which is every edge replaced by a symmetric arc.

One of the topics in graph theory is to determine the eccentric digraph of a graph. The eccentric digraph of a graph was initially introduced by Fred Buckley (Boland and Miller [1]). Some authors have investigated the problem of finding the eccentric digraph. For example, Boland, et.al [2] discussed eccentric digraphs, Gimbert, et.al [4] found the characterisation of the eccentric digraphs, while Boland and Miller [1] determined the eccentric digraph of a digraph. Boland and Miller [1] also proposed an open problem to find the eccentric digraph of various classes of graphs. In this paper we tackle this open problem for the class of a lintang graph.

2. Main Results

According to Nurdin, et al. [5], a lintang graph, denoted by \(L_m\), is defined as \(L_m = \overline{K_2 + K_m}\), for \(m \geq 1\). We assume that the lintang graph has vertex set \(V_1 = \{u_1, u_2\}\) as polar vertices and \(V_2 = \{v_1, v_2, \ldots, v_m\}\) as lintang vertices. While the edge set \(E(L_m) = \{e_1, e_2, \ldots, e_{2m}\}\), where the edge \(e_i = v_i u_1\) and \(e_i + m = v_i u_2\) for every \(i = 1, 2, \ldots, m\).

The following steps are to determine the eccentric digraph of lintang graph \(L_m\) for \(m = 1\) and \(m > 1\).

The first step, we determined the distance from vertex \(u\) to any vertex \(v\) in the graph, denoted by \(d(u, v)\), using Breadth First Search (BFS) Moore Algorithm taken from Chartrand and Oellermann [3] as follows.

1. Take any vertex, say \(u\), and labeled 0 stating the distance from \(u\) to itself , and other vertices are labeled \(\infty\)
2. All vertices having label \(\infty\) adjacent to \(u\) are labeled by 1
3. All vertices having label \(\infty\) adjacent to 1 are labeled by 2 and so on until the required vertex, say \(v\), has already labeled.

The second step, we determined the vertex eccentricity \(u\) by choosing the maximum distance from the vertex \(u\), and so we obtained the eccentric vertex \(v\) from \(u\) if \(d(u, v) = e(u)\).

The final step, by joining an arc from vertex \(u\) to its eccentric vertex, so we obtained the eccentric digraph from the given graph.
Theorem 1: The eccentric digraph of a lintang graph $L_1$, $ED(L_1)$, is a digraph cycle $C_3$ with the set of vertices $V(ED(L_1)) = \{u_1, u_2, v_1\}$ and the set of arcs $A(ED(L_1)) = \{v_1 u_j, u_i u_j \mid i, j = 1, 2 \text{ and } i \neq j\}$.

Proof: Let $L_1$ be a lintang graph with the polar vertices $u_1$ and $u_2$, while its lintang vertex is vertex $v_1$. After applying BFS Moore Algorithm, it is obtained that the eccentricity of lintang vertex $v_1$ is 1 or $e(v_1) = 1$ with its eccentric vertices are all polar vertices $u_j, j = 1, 2$, and so vertex $v_1$ is joined to vertex $u_j$ by an arc $v_1 u_j$. On the other hand, the eccentricity of polar vertex $u_j, j = 1, 2$ is equal to 2 or $e(u_j) = 2$ where its eccentric vertex is its other polar vertex $u_i, j = 1, 2$ and $i \neq j$. So, the vertex $u_i$ is joined to $u_j$ by an arc $u_i u_j$, as required.

The following Figure 1 is to describe a lintang graph $L_1$ and its eccentric digraph.

![Figure 1](image)

Figure 1. Lintang Graph $L_1$ and Its Eccentric Digraph

Theorem 2: The eccentric digraph of a lintang graph $L_m$, $m > 1$, is $ED(L_m) = \text{digraph } K_m \cup K_2$ with the set of vertices $V(ED(L_m)) = \{u_1, u_2, v_1, v_2, \ldots, v_m\}$ and the set of arcs $A(ED(L_m)) = \{u_i u_j \mid i, j = 1, 2 \text{ and } i \neq j\}$ and $A(ED(L_m)) = \{v_i v_j \mid i, j = 1, 2, \ldots, m \text{ and } i \neq j\}$.

Proof: By applying BFS Moore algorithm, it is obtained that the eccentricity for all vertices of lintang graph $L_m$, $m > 1$, is 2. The eccentric vertices of lintang vertices $v_i, i = 1, 2, \ldots, m$, are lintang vertices $v_j, j = 1, 2, \ldots, m$ and $i \neq j$. So, vertex $v_i$ is joined to vertex $v_j$ by an arc $v_i v_j$. The eccentric vertex of polar vertex $u_i$ is a polar vertex $u_j$, for every $i, j$ and $i \neq j$. So, vertex $u_i$ is joined to vertex $u_j$ by an arc $u_i u_j$. By observation, it is easy to check that $ED(L_2) = \text{digraph } K_2 \cup K_2$, $ED(L_3) = \text{digraph } K_3 \cup K_2$, and $ED(L_4) = \text{digraph } K_4 \cup K_2$. The lintang graphs $L_2$, $L_3$ and $L_4$ are described in Figure 2.
By considering this pattern, it is easy to see that $ED(L_m)$, $m > 1$, is a digraph $K_m \cup K_2$ which is the complement of lintang graph $L_m$ consisting the set of vertices $V(ED(L_m)) = \{u_1, u_2, v_1, v_2, \ldots, v_m\}$ and the set of arcs $A(ED(L_m)) = \{u_i u_j \mid i, j = 1, 2 \text{ and } i \neq j\}$ and $A(ED(L_m)) = \{v_i v_j \mid i, j = 1, 2, \ldots, m \text{ and } i \neq j\}$, as required.

3. Concluding Remarks

As mentioned in previous sections the main goal of this paper is to find the eccentric digraph of a graph. A few authors have conducted research on this such problems. Most of them have left some open problems on their paper for the future research. We suggest readers to investigate the problem proposed by Boland and Miller [1] by considering other classes of graphs.
References


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