



THE ECCENTRIC DIGRAPH OF (n, t) -KITE GRAPH

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Abstract

Let G be a graph with a set of vertices $V(G)$ and a set of edges $E(G)$. The distance from vertex u to vertex v in G is the length of the shortest path from vertex u to v . The eccentricity $e(u)$ of a vertex u is the maximum distance of u to any other vertices of G . A vertex v is an eccentric vertex of vertex u if the distance from u to v is equal to $e(u)$. The eccentric digraph $ED(G)$ of a graph G is the digraph that has the same set of vertices as G , and there is an arc (directed edge) from u to v in $ED(G)$ if and only if v is an eccentric vertex of u in G .

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2010 Mathematics Subject Classification: 05C20.

Keywords and phrases: eccentric digraph, eccentricity, (n, t) -kite.

Submitted by K. K. Azad

Received November 29, 2012

In this paper, we determine the eccentric digraph of an (n, t) -kite graph.

I. Introduction

The notations and terminologies mostly follow that of Chartrand and Oellermann [2] and Gallian [3]. Let G be a graph with a set of vertices $V(G)$ and a set of edges $E(G)$. The distance from vertex u to vertex v in G , denoted by $d(u, v)$, is the length of the shortest path from vertex u to v . If there is no path joining vertex u and vertex v , then $d(u, v) = \infty$. The eccentricity of vertex u in graph G is the maximum distance from vertex u to any other vertices in G , denoted by $e(u)$, and so $e(u) = \max\{d(u, v) | v \in V(G)\}$. Radius of a graph G , denoted by $rad(G)$, is the minimum eccentricity of every vertex in G . The diameter of a graph G , denoted by $diam(G)$, is the maximum eccentricity of every vertex in G . If $e(u) = rad(G)$, then vertex u is called *central vertex*. Center of a graph G , denoted by $cen(G)$, is an induced subgraph formed from central vertices of G . Vertex v is an eccentric vertex from u if $d(u, v) = e(u)$. The eccentric digraph $ED(G)$ of a graph G is a graph that has the same set of vertices as G , $V(ED(G)) = V(G)$, and there is an arc (directed edge) joining vertex u to v if v is an eccentric vertex from u . An arc of a digraph D joining vertex u to v and vertex v to u is called a *symmetric arc*.

Boland and Miller [1] proposed an open problem to find the eccentric digraph of various classes of graphs. Some authors have investigated the problem of finding the eccentric digraph. For example, Gimbert et al. [4] found the characterisation of the eccentric digraphs while Wang and Sun [7] found the eccentric digraphs of the digraphs. Kusmayadi and Sudibyoy [5] also found the eccentric digraph of friendship graph and firecracker graph. In this paper, we tackle the open problem proposed by Boland and Miller [1]. In particular, we determine the eccentric digraph of an (n, t) -kite graph.

II. The Eccentric Digraph of (n, t) -kite Graph

According to Wallis [6], an (n, t) -kite graph, or $k_{n,t}$ for short, consists of a cycle of length n with a t -edge path (the tail) attached to one vertex. We assume that the (n, t) -kite graph has vertex set $V(k_{n,t}) = \{x_1, x_2, \dots, x_t, y_0, y_1, \dots, y_{n-1}\}$. The (n, t) -kite graph can be described as in Figure 1.

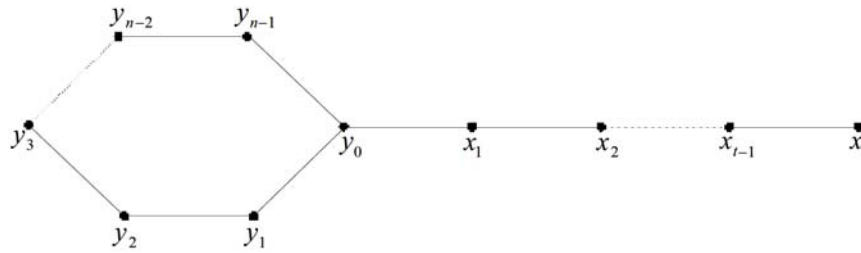


Figure 1. The (n, t) -kite graph $k_{n,t}$.

The following results are the eccentric digraphs of (n, t) -kite graph. We divide into some cases according to the different values of n and t .

Theorem 1. Let $k_{n,t}$ be (n, t) -kite graph, for $t = \lfloor \frac{n}{2} \rfloor$, $n \geq 3$, $t \geq 1$.

Then the eccentric digraphs $ED(k_{n,t})$ are 5-partite digraph $F_{1,1,2,t-1,n-3}$, for n odd, and 5-partite digraph $F_{1,1,1,t-1,n-2}$, for n even.

Proof. We consider two cases according to the values of n .

Case 1. n odd.

By determining the eccentricity and eccentric vertex for each vertex of the (n, t) -kite graph, it is easy to check that the arcs are from vertex y_0 to the vertex x_t and y_i for $i = \frac{n-1}{2}, \frac{n+1}{2}$. Also, the arcs are from vertex y_i to the vertex x_t for $i \in [1, n]$. In addition, the arcs are from vertex x_j to the vertex y_i for $j \in [1, t]$ and $i = \frac{n-1}{2}, \frac{n+1}{2}$. Now, the arcs are from vertex

x_t to the vertex y_i for $i = \frac{n-1}{2}, \frac{n+1}{2}$ and so not all arcs are symmetric.

Based on these arcs, the vertex set $V(ED(k_{n,t}))$ can be partitioned into

five subsets of vertices $V_1 = \{x_t\}$, $V_2 = \left\{y_{\frac{n-1}{2}}, y_{\frac{n+1}{2}}\right\}$, $V_3 = \{y_0\}$,

$V_4 = \{x_1, x_2, \dots, x_{t-1}\}$ and

$$V_5 = \left\{y_1, y_2, \dots, y_{\frac{n-3}{2}}, y_{\frac{n+3}{2}}, y_{\frac{n+5}{2}}, \dots, y_{n-1}\right\}.$$

All arcs from vertices of V_4 are incident to the vertices of V_2 , while all arcs from vertices of V_5 are incident to the vertices of V_1 , and all arcs from vertices of V_3 are incident to the vertices of V_1 and V_2 . The arcs from V_1 and V_2 are symmetric arcs. Hence the eccentric digraph of $k_{n,t}$ is a 5-partite digraph $F_{1,1,2,t-1,n-3}$.

Case 2. n even.

By determining the eccentricity and vertex eccentric for each vertex of the (n, t) -kite graph, it is easy to check that the arcs are from vertex y_0 to the vertex x_t and to the vertex $y_{\frac{n}{2}}$. Also, the arcs are from vertex y_i to the

vertex x_t for $i \in [0, n]$. In addition, the arcs are from vertex x_j to the

vertex y_i for $j \in [1, t]$ and $i = \frac{n-1}{2}, \frac{n+1}{2}$. Now, the arcs are from vertex

x_t to the vertex $y_{\frac{n}{2}}$ and so not all arcs are symmetric. Based on these arcs,

the vertex set $V(ED(k_{n,t}))$ can be partitioned into five subsets of vertices

$V_1 = \{x_t\}$, $V_2 = \left\{y_{\frac{n}{2}}\right\}$, $V_3 = \{y_0\}$, $V_4 = \{x_1, x_2, \dots, x_{t-1}\}$ and

$$V_5 = \left\{y_1, y_2, \dots, y_{\frac{n-2}{2}}, y_{\frac{n+2}{2}}, y_{\frac{n+4}{2}}, \dots, y_{n-1}\right\}.$$

All arcs from vertices of V_4 are incident to the vertices of V_2 , while all arcs from vertices of V_5 are incident to the vertices of V_1 , and all arcs from vertices of V_3 are incident to the vertices of V_1 and V_2 . The arcs from V_1 and V_2 are symmetric arcs. From these partitions, then there is no arc from the same subsets. Therefore, the digraph can be formed to be 5-partite digraph $F_{1,1,1,t-1,n-2}$. \square

For simplicity, let $z = \left\lfloor \frac{\left\lfloor \frac{n}{2} \right\rfloor + t}{2} \right\rfloor$ for $t > \frac{n}{2}$.

Theorem 2. Let $k_{n,t}$ be (n, t) -kite graph, for $t > \frac{n}{2}$, $n \geq 3$. Then the eccentric digraph $ED(k_{n,t})$ is

- a. 5-partite digraph $F_{1,1,1,z-1,n+t-z-2}$ for n even and $\frac{n}{2} + t$ even,
- b. 4-partite digraph $F_{1,1,z,n+t-z-2}$ for n even and $\frac{n}{2} + t$ odd,
- c. 5-partite digraph $F_{1,2,1,z-1,n+t-z-3}$ for n odd and $\left\lfloor \frac{n}{2} \right\rfloor + t$ even,
- d. 4-partite digraph $F_{1,2,z,n+t-z-3}$ for n odd and $\left\lfloor \frac{n}{2} \right\rfloor + t$ odd.

Proof. We consider four cases according to the values of n and t .

Case 1. n even and $\frac{n}{2} + t$ even.

By determining the eccentricity and eccentric vertex for each vertex of the (n, t) -kite graph, we observe that the arcs are from vertex y_i to the vertex x_t for $i \in [0, n - 1]$. Also, the arcs are from vertex x_i to the vertex $y_{\frac{n}{2}}$ for $i \in [t - z, t]$. In addition, the arcs are from vertex x_i to the vertex

x_t for $i \in [i, t - z]$ and so not all arcs are symmetric. Based on these arcs, the vertex set $V(ED(k_{n,t}))$ can be partitioned into five subsets of vertices

$$V_1 = \{x_t\}, V_2 = \left\{y_{\frac{n}{2}}\right\}, V_3 = \{x_{t-z}\}, V_4 = \{x_{t+1-z}, \dots, x_{t-1}\} \text{ and}$$

$$V_5 = \left\{y_0, \dots, y_{\frac{n-2}{2}}, y_{\frac{n+2}{2}}, \dots, y_{n-1}, x_1, \dots, x_{t-z-1}\right\}.$$

All arcs from vertices of V_4 are incident to the vertices of V_2 , while all arcs from vertices of V_5 are incident to the vertices of V_1 , and all arcs from vertices of V_3 are incident to the vertices of V_1 and V_2 . The arcs from V_1 and V_2 are symmetric arcs. From these partitions, then there is no arc from the same subsets. Therefore, the digraph can be formed to be 5-partite digraph $F_{1,1,1,z-1,n+t-z-2}$.

Case 2. n even and $\frac{n}{2} + t$ odd.

By determining the eccentricity and eccentric vertex for each vertex of the (n, t) -kite graph, we observe that the arcs are from vertex y_i to the vertex x_t for $i \in [0, n - 1]$. Also, the arcs are from vertex x_i to the vertex $y_{\frac{n}{2}}$ for $i \in [t - z, t]$. In addition, the arcs are from vertex x_i to the vertex x_t for $i \in [i, t - z - 1]$ and so not all arcs are symmetric. Based on these arcs, the vertex set $V(ED(k_{n,t}))$ can be partitioned into four subsets of

$$\text{vertices } V_1 = \{x_t\}, V_2 = \left\{y_{\frac{n}{2}}\right\}, V_3 = \{x_{t-z}, \dots, x_{t-1}\} \text{ and}$$

$$V_4 = \left\{y_0, \dots, y_{\frac{n-2}{2}}, y_{\frac{n+2}{2}}, \dots, y_{n-1}, x_1, \dots, x_{t-z-1}\right\}.$$

All arcs from vertices of V_4 are incident to the vertices of V_1 , while all arcs from vertices of V_3 are incident to the vertices of V_2 . The arcs from V_1 and V_2 are symmetric arcs. From these partitions, then there is no arc from the same subsets. Therefore, the digraph can be formed to be 4-partite digraph $F_{1,1,z,n+t-z-2}$.

Case 3. n odd and $\left\lfloor \frac{n}{2} \right\rfloor + t$ even.

By determining the eccentricity and eccentric vertex for each vertex of the (n, t) -kite graph, it is easy to check that the arcs are from vertex y_i to the vertex x_t for $i \in [0, n - 1]$. Also, the arcs are from vertex x_i to the vertex y_j for every $i \in [t - z, t]$ and $j = \frac{n-1}{2}, \frac{n+1}{2}$. In addition, the arcs are from vertex x_i to the vertex x_t for $i \in [i, t - z]$ and so not all arcs are symmetric. Based on these arcs, the vertex set $V(ED(k_{n,t}))$ can be partitioned into five subsets of vertices $V_1 = \{x_t\}$, $V_2 = \left\{ \frac{y_{n-1}}{2}, \frac{y_{n+1}}{2} \right\}$, $V_3 = \{x_{t-z}\}$, $V_4 = \{x_{t+1-z}, \dots, x_{t-1}\}$ and

$$V_5 = \left\{ y_0, \dots, \frac{y_{n-3}}{2}, \frac{y_{n+3}}{2}, \dots, y_{n-1}, x_1, \dots, x_{t-z-1} \right\}.$$

All arcs from vertices of V_4 are incident to the vertices of V_2 , while all arcs from vertices of V_5 are incident to the vertices of V_1 , and all arcs from vertices of V_3 are incident to the vertices of V_1 and V_2 . The arcs from V_1 and V_2 are symmetric arcs. From these partitions, then there is no arc from the same subsets. Therefore, the digraph can be formed to be 5-partite digraph $F_{1,2,1,z-1,n+t-z-3}$.

Case 4. n odd and $\left\lfloor \frac{n}{2} \right\rfloor + t$ odd.

Proof. By determining the eccentricity and eccentric vertex for each vertex of the (n, t) -kite graph, we observe that the arcs are from vertex y_i to the vertex x_t for $i \in [0, n - 1]$. Also, the arcs are from vertex x_i to the vertex y_j for $i \in [t - z, t]$ and $j = \frac{n - 1}{2}, \frac{n + 1}{2}$. In addition, the arcs are from vertex x_i to the vertex x_t for $i \in [i, t - z - 1]$ and so not all arcs are symmetric. Based on these arcs, the vertex set $V(ED(k_{n,t}))$ can be

$$\text{partitioned into four subsets of vertices } V_1 = \{x_t\}, V_2 = \left\{ y_{\frac{n-1}{2}}, y_{\frac{n+1}{2}} \right\},$$

$$V_3 = \{x_{t-z}, \dots, x_{t-1}\} \text{ and } V_4 = \left\{ y_0, \dots, y_{\frac{n-3}{2}}, y_{\frac{n+3}{2}}, \dots, y_{n-1}, x_1, \dots, x_{t-z-1} \right\}.$$

All arcs from vertices of V_4 are incident to the vertices of V_1 , while all arcs from vertices of V_3 are incident to the vertices of V_2 . The arcs from V_1 and V_2 are symmetric arcs. From these partitions, then there is no arc from the same subsets. Therefore, the digraph can be formed to be 4-partite digraph $F_{1,2,z,n+t-z-3}$. □

Theorem 3. Let $k_{n,t}$ be (n, t) -kite graph, for $t < \frac{n}{2}, n \geq 3$. Then the eccentric digraph of $k_{n,t}$, $ED(k_{n,t})$ is a digraph having vertex set $V(k_{n,t})$ and the arc set are

$$\left\{ \overrightarrow{x_i y_j}, i \in [1, t], j = \frac{n - 1}{2}, \frac{n + 1}{2} \right\}$$

$$\cup \left\{ \overrightarrow{y_l y_m}, l \in \left[0, \frac{n - 2t - 1}{2} \right], m = \frac{n + 2l - 1}{2}, \frac{n + 2l + 1}{2} \right\}$$

$$\cup \left\{ \overrightarrow{y_p x_t}, p \in \left[\frac{n - 2t - 1}{2}, \frac{n + 2t + 1}{2} \right] \right\}$$

$$\cup \left\{ \overrightarrow{y_q y_r}, q \in \left[\frac{n+2t+1}{2}, n-1 \right], \right.$$

$$\left. r = \left(\frac{n+2p-1}{2} \right) \bmod n, \left(\frac{n+2p+1}{2} \right) \bmod n \right\} \text{ for } n \text{ odd} \quad (1)$$

and

$$\left\{ \overrightarrow{x_i y_n}, i \in [1, t] \right\}$$

$$\cup \left\{ \overrightarrow{y_j y_l}, j \in \left[0, \frac{n-2t}{2} \right], l = \frac{n+2j}{2} \right\} \cup \left\{ \overrightarrow{y_m x_t}, m = \frac{n-2t}{2}, \frac{n+2t}{2} \right\}$$

$$\cup \left\{ \overrightarrow{y_p y_q}, p \in \left[\frac{n+2t}{2}, n-1 \right], q = \left(\frac{n+2p}{2} \right) \bmod n \right\} \text{ for } n \text{ even.} \quad (2)$$

Proof. By determining the eccentricity and eccentric vertex for each vertex of the (n, t) -kite graph, it is easy to check that the arcs are as stated in (1) for n odd and (2) for n even. □

Acknowledgement

The authors would like to thank to Sebelas Maret University, Surakarta, Indonesia for funding this fundamental research scheme in 2012.

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