On γ-Labeling of (*n*,*t*)-Kite Graph

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Abstract

Let G(V,E) be a graph of order n and size m. A γ-labeling of G is an one-to-one function $f: V(G) \rightarrow \{0, 1, 2, \ldots, m\}$ *that induces a labeling* $f': E(G) \rightarrow \{1, 2, 3, \ldots, m\}$ *of the edges of G defined by* $f'(e) = |f(u)-f(v)|$ *for each edge e = uv of G. The value of a γ-labeling f is denoted by* $val(f) = \Sigma_{e\varepsilon E} f'(e)$. *The maximum value of a γ-labeling of G is defined by* $val_{max}(G) = max\{val(f) : f \text{ is a } \gamma\text{- labeling of } G\}$, *while the minimum value of a γ-labeling of G is defined by* $val_{min}(G) = min\{val(f) : f \text{ is a } \gamma\text{- labeling of } G\}$. *In this paper we investigate the* $val_{min}(G)$ *of an (n,t)-kite graph G for every integer* $n \geq 3$, *and the lower bound of the* $val_{max}(G)$ *of an (n,t)-kite graphs G for n =3 and n=4.*

Keywords: γ-labeling, (n,t)-kite graphs, Maximum value, Minimum value.

Abstrak

Misal G(V,E) adalah graf dengan banyak titik n dan banyak sisi m. Suatu pelabelan-γ pada graf G adalah fungsi satu-satu $f: V(G) \rightarrow \{0, 1, 2, \ldots, m\}$ *yang menghasilkan pelabelan* $f': E(G) \rightarrow \{1, 2, 3, \ldots, m\}$ *pada sisi-sisi dari G yang didefinisikan oleh* $f'(e) = |f(u)-f(v)|$ *untuk setiap sisi e = uv pada G. Nilai dari pelabelan-γ f dilambangkan dengan* $val(f) = \Sigma_{e\varepsilon E} f'(e)$. *Nilai maksimum untuk pelabelan-γ f dari graf G didefinisikan oleh* $val_{max}(G) = max\{val(f) : f \text{ adalah pelabelan -}\gamma \text{ dari } G\}$, *sedangkan nilai minimum untuk pelabelan-γ f dari G didefinisikan oleh* $val_{min}(G) = min\{val(f) : f \text{ adalah pelabelan -}\gamma \text{ dari } G\}$. *Pada artikel ini kami memberikan* $val_{min}(G)$ *dari graf (n,t)-kite G untuk sembarang bilangan bulat* $n \geq 3$, *dan batas bawah untuk* $val_{max}(G)$ *dari graf (n,t)-kite G untuk n=3 dan n=4.*

Kata kunci: Pelabelan-γ, Graf (n,t)-kite, Nilai maksimum, Nilai minimum.

1. Introduction

Let *G(V,E)* be a finite, connected, simple and undirected graph, and let *V* and *E* denote the vertex set and edge set of *G*, respectively. Wallis (2001) defined a labeling (or valuation) of graph as follows: a labeling of a graph is a map that carries graph elements to the numbers (usually to the positive or non negative integers). The most common choices of domain are the set of all vertices and edges (such a labeling is called a total labeling), the vertex set alone (vertex labeling), or the edge set alone (edge labeling).

The graph labeling can also be defined as different function (Gallian, 2011). For a graph *G* of order *n* and size *m*, a γ-labeling of *G* is an one-to-one function $f: V(G) \rightarrow \{0, 1, 2, \ldots m\}$ that induces a labeling $f': E(G) \rightarrow \{1, 2, 3, \ldots, m\}$ of the edges of *G* defined by $f'(e) = | f(u) – f(v) |$ for each edge $e = uv$ of *G*. Chartrand *et al.* (2005) showed that every connected graph has a γ-labeling. Each γ-labeling *f* of graph *G* of order *n* and size *m* is assigned a value denoted by *val(f)* and defined by $val(f)=\Sigma_{e\varepsilon E} f'(e)$. Since *f* is one-to-one function from *V(G)* to {0, 1, 2, …, *m*}, it follows that $f'(e) \geq 1$ for each edge *e* in *G* and so $val(f) \geq m$. The maximum value of a γ-labeling of *G* is defined by $val_{max}(G)=max\ \{val(f): f \text{ is a } \gamma\text{-labeling of } G\}$, while the minimum value of *G* is defined by $val_{min}(G) = min\ \{val(f): f \text{ is a } \gamma\text{-labeling of } G\}$. Chartrand *et al.* (2005) observed the γ-labeling of some well-known classes of graphs *G* such as paths P_n, stars $K_{1,n-1}$ and cycles C_n. They also determined the value $val_{max}(G)$ and $val_{min}(G)$ of these graphs. For path P_n, $val_{max}(P_n) \geq \left\lfloor \frac{n^2-2}{2} \right\rfloor$, for $n \geq 2$ and $val_{min}(P_n) = n - 1$. For cycle C_n, $val_{max}(C_n) = \frac{(n-1)(n+3)}{2}$ for every odd integer $n \geq 3$ and $val_{max}(C_n) = \frac{n(n+2)}{2}$ for an even integer $n \geq 4$, while $val_{min}(C_n) = 2(n-1)$. For star $K_{1,n-1}$, $val_{max}(K_{1,n-1}) = \binom{n}{2}$ and $val_{min}(K_{1,n-1}) = \binom{\left\lfloor \frac{n+1}{2} \right\rfloor}{2} + \binom{\left\lceil \frac{n+1}{2} \right\rceil}{2}$. Roswitha *et al.* (2007) determined the value of $val_{max}(G)$ and $val_{min}(G)$ of the γ-labeling of Petersen 3-regular graph. Indriati *et al.* (2009a, 2009b, 2010) already worked on double-stars, firecrackers, *n*-sun graphs, wheels, fans, helms and flowers for $n \geq 3$. In this paper we consider the γ-labeling of an (*n*,*t*)-kite graphs. We also determine the val_{min} for $n \geq 3$ and the lower bound of the val_{max} of their γ-labeling for *n*=3 and *n*=4.

2. Main Results

In this section, we present the value of the γ-labeling of an (*n*,*t*)-kite graph. First we establish a formula for the val_{min} of the γ-labeling of an (*n*,*t*)-kite graph for $n\geq3$, then the formula for the lower bound of the val_{max} of the γ-labeling for n=3 and n=4. Wallis (2001) defined this graph as follows: an (n,t)-kite graph, or G for short, consists of a cycle of length n with a t-edge path (the tail) attached to one vertex (see Figure 1).

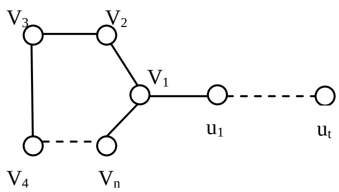

Figure 1. The (n,t)-kite graph.

2.1 The minimum value of a γ-labeling

Theorem 2.1 determines the minimum value of a γ-labeling of an (n,t)-kite graph G for $n\geq3$ and every positive integer t, and also introduces a formula how to construct the labels of all vertices.

Theorem 2.1. For every integer $n\geq3$,

$$val_{\min}(G) = 2(n-1) + t.$$

Proof. The (n,t)-kite graph G has a cycle of length n, and t-edge path. Suppose the n vertices of the cycle are $\{v_1, v_2, \ldots, v_n\}$ and the t vertices of the t-edge path (tail) are $\{u_1, u_2, \ldots, u_t\}$ with v_1 adjacent to u_1. Let f be a γ-labeling of G. Using the result of Chartrand et al. (2005) for the labeling of the cycle, we have

$$val_{\max}(C_n) = (2n-1)$$

with the label of $v_1, v_2, \ldots, v_n$ is n-1, 0, 1, 2, …, n-2 respectively. The t vertices of the tail are labelled by

$$f(u_i) = \begin{cases} n, & \text{if } i = 1 \\ n+i-1, & \text{if } i = 2,3,\ldots,t \end{cases}$$

By this labelling, we obtain

$|f(u_1)-f(v_1)| = |n-n+ 1| = 1$

$|f(u_i)-f(u_{i+1})| = |n+i-1-(n+i+1-1)| = 1$, for i = 1, 2, …, $t-1$.

So, each $e\in E$ of the *t-edge path* has $f'(e)$=1, the minimum label of the edge, and

$$val_{\min}(t\text{-edge path}) = t$$

Therefore, $val_{\min}(G) = val_{min}(C_n) + val_{\min}(t\text{-edge path}) = 2(n-1) + t$. ■

Figure 2 shows that a (4,3)-kite has a $val_{min}(G)$=9, with the vertex label is written near each vertex.

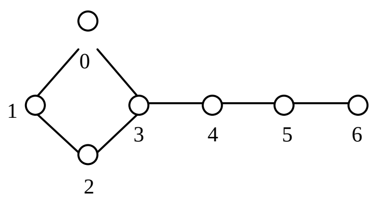

Figure 2. Labeling of a (4,3)-kite graph and its val_{min}.

2.2 The lower bound of the maximum value of a γ-labeling

Now consider the two theorems below. The first theorem (Theorem 2.2) determines the lower bound of the maximum value of a γ-labeling of an (n,t)-kite graph G for n=3, while the second theorem (Theorem 2.3) determines the same matter for n=4. Also, in these theorems we introduce a formula of each labeling.

Theorem 2.2. Let G be an $(3,t)$-kite graph. Then, $val_{max}(G)\geq 2m + mt - \binom{t+1}{2}$.

Proof. The $(3,t)$-kite graph G has a cycle of length 3, and t-edge path. Suppose the three vertices of the cycle are $\{v_1, v_2, v_3\}$ and the t vertices of the t-edge path (tail) are $\{u_1, u_2, \ldots, u_t\}$ with v_1 adjacent to u_1. Let f be a γ-labeling of G. This graph has size of m=3 + t. the proof is divided into two cases, for t is odd and t is even.

a. For t is odd.

The vertices of the cycle are labeled with

$$f(v_i) = \begin{cases} 0 & ,if\ i = 1 \\ m & ,if\ i = 2 \\ \dfrac{m-2}{2} & ,if\ i = 3 \end{cases}$$

So, for the cycle we obtain

$$|f(v_i)-f(v_{i+1})_{mod\ 3}| = |f(v_1)-f(v_2)| + |f(v_2)-f(v_3)| + |f(v_3)-f(v_1)|$$

$$= m + \left|m - \frac{m-2}{2}\right| + \left|\frac{m-2}{2} - 0\right|$$

$$= 2m \qquad (1)$$

The vertices of the tail are labeled with

$$f(u_i) = \begin{cases} m - \dfrac{i+1}{2} & \text{, if } i \text{ is odd, } i \leq t \\ \dfrac{i}{2} & \text{, if } i \text{ is even, } i \leq t-1 \end{cases}$$

For the tail we obtain

$$|f(u_1) - f(v_1)| + \sum_{i=1}^{t-1} |f(u_i) - f(u_{i+1})|$$

$$= (m-1) + (m-2) + (m-3) + \cdots + \left(m - \frac{t+1}{2} - \frac{t-1}{2}\right)$$

$$= (m-1) + (m-2) + (m-3) + \cdots (m-t)$$

$$= mt - \binom{t+1}{2} \qquad (2)$$

Since from (1) and (2), we obtain

$$val_{max}(G)\geq 2m + mt - \binom{t+1}{2}$$

b. For t is even.

The vertices of the cycle are labeled with

$$f(v_i)=\begin{cases}0 & \text{, if } i=1\\ m & \text{, if } i=2\\ \frac{m-1}{2} & \text{, if } i=3\end{cases}$$

So, for the cycle we obtain

$|f(v_i)-f(v_{i+1})_{mod\ 3}| = |f(v_1)-f(v_2)| + |f(v_2)-f(v_3)| + |f(v_3)-f(v_1)|$

$$= m+m-\frac{m-1}{2}+\frac{m-1}{2}-0$$

$$= 2m$$

The vertices of the tail are labeled with

$$f(u_i)=\begin{cases}m-\frac{i+1}{2} & \text{, if } i \text{ is odd, } i\le t-1\\ \frac{i}{2} & \text{, if } i \text{ is even, } i\le t\end{cases}$$

Furthermore, for t is odd, we have

$$|f(u_1)-f(v_1)|+\sum_{i=1}^{t-1}|f(u_i)-f(u_{i+1})|$$

$$= mt-\binom{t+1}{2}$$

Therefore, we obtain

$$val_{\max}(G)\ge 2m+mt-\binom{t+1}{2} \quad \blacksquare$$

Next, we continue for n=4.

Theorem 2.3. Let G be an (4,t)-kite graph. Then, $val_{max}(G)\geq 4m + mt - \binom{t+3}{2}$.

Proof. The (4,t)-kite graph G has a cycle of length 4, and t-edge path. Suppose the four vertices of the cycle are $\{v_1, v_2, v_3, v_4\}$ and the t vertices of the t-edge path (tail) are $\{u_1, u_2, \ldots, u_t\}$ with v_1 adjacent to u_1. Let f be a γ-labeling of G. This graph has size of $m = 4 + t$. Like Theorem 2.2, we prove the two cases of this theorem, for t is odd and t is even.

a. For t is odd.

The vertices of the cycle are labeled with

$$f(v_i)=\begin{cases}0 & \text{, if } i=1\\ m & \text{, if } i=2\\ \frac{t+1}{2} & \text{, if } i=3\\ m-1 & \text{, if } i=4\end{cases}$$

Thus, we obtain that

$|f(v_i)-f(v_{i+1})_{mod\ 4}| = |f(v_1)-f(v_2)| + |f(v_2)-f(v_3)| + |f(v_3)-f(v_4)| + |f(v_4)-f(v_1)|$

$$= m + m\ \frac{t+1}{2} + m-1-\frac{t+1}{2} + m-1$$

$$= 4m- t - 3 \qquad (3)$$

The vertices of the tail are labeled with

$$f(u_i)=\begin{cases}m-\frac{i+3}{2} & \text{, if } i \text{ is odd, } i\le t\\ \frac{i}{2} & \text{, if } i \text{ is even, } i\le t-1\end{cases}$$

So, for the tail we obtain

$$|f(u_1)-f(v_1)|+\sum_{i=1}^{t-1}|f(u_1)-f(u_{i+1})|$$

$$=(m-2)+(m-3)+(m-4)+\ldots+(m-\frac{t+3}{2}-\frac{t-1}{2})$$

$$= mt-\binom{t+3}{2}+3+t \qquad (4)$$

From (3) and (4), we obtain

$$val_{max}(G)\geq 4m + mt - \binom{t+3}{2} \qquad (5)$$

b. For t is even.

The vertices of the cycle are labeled with

$$f(v_i)=\begin{cases}0 & \text{, if } i=1\\ m & \text{, if } i=2\\ 1 & \text{, if } i=3\\ m-1 & \text{, if } i=4\end{cases}$$

So, for the cycle we obtain

$|f(v_i)-f(v_{i+1})_{mod\ 4}| = |f(v_1)-f(v_2)| + |f(v_2)-f(v_3)| + |f(v_3)-f(v_4)| + |f(v_4)-f(v_1)|$

$$= m + m - 1 + m - 2 + m - 1$$

$$= 4m-4 \qquad (6)$$

The vertices of the tail are labeled with

$$f(u_i)=\begin{cases}m-\frac{i+3}{2} & \text{, if } i \text{ is odd, } i\le t-1\\ \frac{i+2}{2} & \text{, if } i \text{ is even, } i\ \le t\end{cases}$$

So, for the tail we obtain

$$|f(u_1)-f(v_1)|+\sum_{i=1}^{t-1}|f(u_i)-f(u_{i+1})|$$

$$= m(-2)+(m-4)+(m-5)+\ldots+\left(m-\frac{t-2}{2}-\frac{t+2}{2}\right)$$

$$= mt-(1+2+3+\cdots+(t+2))+4$$

$$= mt-\binom{t+3}{2}+4 \qquad (7)$$

From (6) and (7), we obtain

$$val_{\max}(G)\ge 4m+mt-\binom{t+3}{2} \qquad (8)$$

Finally, from (5) and (8), the proof is done. ■

Figure 3 shows that a (4,4)-kite has a $val\ (G)$=43, with the vertex label is written near each vertex.

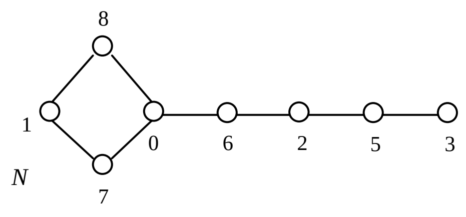

Figure 3. Labeling of a (4,4)-kite graph and its $val(G)$.

Future Work: In this paper we investigate the lower bound of the val_{max} *of a* γ-labeling of an (n,t)-kite for n=3 and n=4. The exact value of the val_{max} of this labeling for any $n\geq 3$ is still open.

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